

# Symmetries of particle motion

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## Abstract

We define affine transport lifts on the tangent bundle by associating a transport rule for tangent vectors with a vector field on the base manifold. The aim is to develop tools for the study of kinetic/dynamical symmetries in relativistic particle motion. The transport lift unifies and generalises the various existing lifted vector fields, with clear geometric interpretations. We find the affine dynamical symmetries of general relativistic particle motion, and compare this to previous results and to the alternative concept of “matter symmetry”.

## 1 Introduction

Vector fields on the tangent bundle  $TM$ , arising as the lifts of vectors or of transformations on the base manifold  $M$ , have been defined and applied in differential geometry, Lagrangian mechanics and relativity. For example the complete (or natural or Lie), horizontal and vertical lifts [1]–[4], the projective and conformal lifts [5] and the matter symmetries of Berezdivin and Sachs [6], [7]. Our aim is to find a more general way of lifting from  $M$  to  $TM$  than the usual definitions that involve only the vector field on  $M$ , and possibly the connection on  $M$ . In fact the matter symmetries of [6] are a step in this direction. We generalise this concept in a way that gives a clear geometric foundation to all the lifts previously defined, and to new lifts which can be defined.

The main idea [8] is to associate a transport rule for tangent vectors with a vector field on  $M$ . This defines a vector field on  $TM$  – the transport lift. The class of *affine transport lifts* (ATL’s) generalises all previously defined lifts in a unified and geometrical way. We find conditions under which ATL’s are dynamical symmetries for particle trajectories in (semi-) Riemannian manifolds.

## 2 Local geometry of the tangent bundle

We give a brief summary of the relevant local differential geometry of the tangent bundle assuming only a knowledge of basic tensor analysis on manifolds. Consider a (semi-) Riemannian  $n$ -manifold  $(M, g)$  with local coordinates  $x^a$  and metric connection  $\Gamma^a_{bc}$  (Christoffel symbols). The tangent bundle  $TM$  is the union of all tangent spaces (fibres)  $T_x M$ ,  $x \in M$ . In relativistic kinetic theory (RKT) the phase space arises out of  $TM$  by restriction to future-directed, non-spacelike tangent vectors [9].

Local coordinates  $x^a$  on  $M$  induce local coordinates  $\xi^I = (x^a, p^b)$  on  $TM$ , where  $p^a$  are the coordinate components of the vector  $p = p^a \partial / \partial x^a$ . Any smooth vector field on  $TM$  can be expressed covariantly via the anholonomic “connection basis”  $\{H_a, V_b\}$  of horizontal and vertical vector fields [4]:

$$H_a = \frac{\partial}{\partial x^a} - \Gamma^b_{ca} p^c \frac{\partial}{\partial p^b}, \quad V_a = \frac{\partial}{\partial p^a}. \quad (2.1)$$

The Lie brackets of the basis vectors are

$$[V_a, V_b] = 0, \quad (2.2)$$

$$[H_a, V_b] = \Gamma^c_{ab} V_c, \quad (2.3)$$

$$[H_a, H_b] = -R^d_{cab} p^c V_d, \quad (2.4)$$

where  $R_{abcd}$  is the Riemann curvature tensor. The vector field

$$\Gamma = p^a H_a \quad (2.5)$$

has integral curves on  $TM$  which are the natural lifts of geodesics on  $M$ .  $\Gamma$  is called the geodesic spray or, in RKT, the Liouville vector field.

For a vector field  $Y = Y^a(x)\partial/\partial x^a$  on  $M$ , various lifted vector fields have been defined on  $TM$ :

$$\text{Horizontal lift} : Y \rightarrow \bar{Y} = Y^a(x)H_a, \quad (2.6)$$

$$\text{Vertical lift} : Y \rightarrow \hat{Y} = Y^a(x)V_a, \quad (2.7)$$

$$\text{Complete lift} : Y \rightarrow \tilde{Y} = Y^a(x)H_a + \nabla_b Y^a(x)p^b V_a, \quad (2.8)$$

$$\text{Iwai's lift} : Y \rightarrow Y^\dagger = \tilde{Y} - 2\psi(x)p^a V_a, \quad (2.9)$$

where  $\nabla$  is the covariant derivative, and  $\psi$  is proportional to  $\nabla_a Y^a$  in (2.9).

We can also define the vertical lift of a rank-2 tensor field [1]

$$A \rightarrow \hat{A} = A^a_b(x)p^b V_a, \quad (2.10)$$

with a special case being the Euler vector field [4]

$$\Delta = \hat{\delta} = p^a V_a. \quad (2.11)$$

Matter symmetries in RKT have been defined [6] in terms of a vector field  $Y$  and a skew rank-2 tensor field  $A$  on  $M$ :

$$(Y, A) \rightarrow Y^a(x)H_a + A^a_b(x)p^b V_a, \quad A_{(ab)} = 0, \quad (2.12)$$

where round brackets enclosing indices denote symmetrisation.

A dynamical system on  $M$  is defined [2], [3] by a congruence of trajectories on  $TM$ . The tangent vector field to these trajectories is the dynamical vector field  $\Gamma$ , e.g. (2.5). A dynamical symmetry is a vector field  $\Sigma$  that maps trajectories into trajectories with possibly rescaled tangent vector field. Thus  $(\exp \varepsilon \mathcal{L}_\Sigma)\Gamma$  is parallel to  $\Gamma$ , where  $\mathcal{L}$  is the Lie derivative. Hence

$$\mathcal{L}_\Sigma \Gamma \equiv [\Sigma, \Gamma] = -\psi \Gamma, \quad (2.13)$$

for some  $\psi(x, p)$ , is the condition for  $\Sigma$  to be a dynamical symmetry. The nature of the rescaling depends on  $\psi(x, p)$ . If  $\psi = \psi(x)$ , then the rescaling is constant on each fibre  $T_x M$ . If  $\psi = 0$ , then there is no rescaling and  $\Sigma$  is said to be a Lie symmetry on  $TM$ .

### 3 Transport lifts

Let  $Y = d/d\sigma$  be a vector field on  $M$  and  $\Lambda$  a smooth local rule governing the transport of tangent vectors along the integral curves of  $Y$ . Thus any  $u^a$  at  $x^a(\sigma)$  is mapped under  $\Lambda$  to  $u'^a$  at  $x'^a = x^a(\sigma + \varepsilon)$ , with  $u'^a = \Lambda^a(x, u; \varepsilon)$ . This defines curves  $(x^a(\sigma), p^b(\sigma))$  in  $TM$ , with

$$\frac{dx^a}{d\sigma} = Y^a(x), \quad \frac{dp^a}{d\sigma} \equiv \lambda^a(x, p) = \frac{\partial \Lambda^a(x, p; 0)}{\partial \varepsilon}. \quad (3.1)$$

We can define a vector field on  $TM$  with integral curves  $(x^a(\sigma), p^b(\sigma))$  given by (3.1). We call this the *transport lift* [8] on  $TM$  of the vector field  $Y$  and of the transport rule  $\Lambda$  along  $Y$ . The transport lift is given locally by

$$(Y, \Lambda) \rightarrow Y^a(x) \frac{\partial}{\partial x^a} + \lambda^a(x, p) \frac{\partial}{\partial p^a} \quad (3.2)$$

$$= Y^a(x)H_a + [\lambda^a(x, p) + \Gamma^a_{bc}(x)p^b Y^c(x)] V_a. \quad (3.3)$$

The transport lift (3.2) combines the point transformations generated by  $Y$  on  $M$  with the tangent vector transformations generated by  $\Lambda$  on  $M$ . In general, the transport rule  $\Lambda$  along  $Y$  is not defined purely by tensor fields on  $M$ . However this is the case for an affine transport rule, for which

$$\Lambda^a(x, u; \varepsilon) = \Omega^a_b(x; \varepsilon)u^b + K^a(x; \varepsilon).$$

Thus the *affine transport lift* (ATL) of  $(Y, \Lambda)$  on  $M$  has the form [8]

$$Y^{(A, k)} = Y^a(x)H_a + [A^a_b(x)p^b + k^a(x)]V_a, \quad (3.4)$$

where

$$A^a_b(x) = \omega^a_b(x) + \Gamma^a_{bc}(x)Y^c(x), \quad (3.5)$$

$$\omega^a_b(x) = \frac{\partial \Omega^a_b(x; 0)}{\partial \varepsilon}, \quad k^a(x) = \frac{\partial K^a(x; 0)}{\partial \varepsilon}. \quad (3.6)$$

It follows that  $k$  is a vector field on  $M$ , whereas  $\omega$  is not a tensor field unless  $Y = 0$ . Furthermore,  $A$  as defined by (3.5) is a tensor field, and the vertical component in (3.4) therefore transforms covariantly. The transport rule  $\Lambda$  is thus covariantly determined by  $A$  and  $k$ .

By (3.4), the integral curves of  $Y^{(A, k)}$  satisfy

$$\frac{dx^a}{d\sigma} = Y^a(x), \quad (3.7)$$

$$\begin{aligned} \frac{dp^a}{d\sigma} &= \omega^a_b(x)p^b + k^a(x) \\ &= [A^a_b(x) - \Gamma^a_{bc}(x)Y^c(x)]p^b + k^a(x). \end{aligned} \quad (3.8)$$

We can rewrite (3.8) as

$$\frac{Dp^a}{d\sigma} = A^a_b p^b + k^a,$$

which shows that  $A$  and  $k$  determine the rate of change of tangent vectors under  $\Lambda$  relative to parallel transport. In the case  $k = 0$ , we get a particularly simple interpretation of  $A$ :

$$A^a_b u^b = \nabla_Y u^a \quad \text{or} \quad A(u) = \nabla_Y u, \quad (3.9)$$

for all  $u$  along  $Y$ . This equation is important for the geometric construction of lifts (see below). The class of *linear transport lifts* (LTL's) arises as the special case  $k^a = 0$ , and we write

$$Y^{(A)} \equiv Y^{(A, 0)}.$$

LTL's encompass all previously defined lifts apart from the vertical lift (2.7).

Now from (3.4) we get

$$\alpha Y^{(A, k)} + \beta Z^{(B, \ell)} = (\alpha Y + \beta Z)^{(\alpha A + \beta B, \alpha k + \beta \ell)}, \quad (3.10)$$

for any scalars  $\alpha, \beta$  on  $M$ . (Note that  $A$  and  $B$  depend, respectively, on  $Y$  and  $Z$ . In particular, this means that in general the taking of the affine transport lift is not a linear operation.) Thus the ATL's form a linear subspace. Furthermore, (2.2–4) give

$$[Y^{(A, k)}, Z^{(B, \ell)}] = [Y, Z]^{(C, m)}, \quad (3.11)$$

where

$$C = \nabla_Y B - \nabla_Z A - [A, B] - R(Y, Z), \quad (3.12)$$

$$m = \nabla_Y \ell - \nabla_Z k - A(\ell) + B(k). \quad (3.13)$$

$C$  is a rank-2 tensor field on  $M$ , with  $[A, B]$  the tensor commutator, and  $R(Y, Z)^a_b = R^a_{bcd}Y^cZ^d$ . By (3.10–13), the ATL's form a Lie algebra. The LTL's are a subalgebra (but not an ideal).

Before limiting ourselves to the linear case, we regain the vertical lift (2.7) of a vector field:

$$\widehat{Z} = 0^{(0,Z)}. \quad (3.14)$$

By (3.4), if  $Y = 0$ , we regain the vertical lift (2.10) of the rank-2 tensor field  $A$ :

$$0^{(A)} = A^a{}_{bp} V_a = \widehat{A}. \quad (3.15)$$

$0^{(A)}$  generates a  $GL(n)$  transformation on each fibre:  $p^a \rightarrow p'^a = (\exp \epsilon A)^a{}_{bp}$ . Thus on each fibre  $T_x M$ ,  $A^a{}_b(x)$  is an element of the Lie algebra  $gl(n)$ . By restricting  $A^a{}_b(x)$  to a particular Lie subalgebra, we see that  $0^{(A)}$  generates gauge transformations of the corresponding Lie group.

In order to regain the horizontal lift (2.6) of a vector field, we require that the transport rule  $\Lambda$  be *parallel transport* along  $Y$  for all  $u$ . When the transport rule  $\Lambda$  is chosen to be *Lie transport* (“dragging along”), we regain the complete lift (2.8):

$$\overline{Y} = Y^{(0)}, \quad \widetilde{Y} = Y^{(\nabla Y)}. \quad (3.16)$$

Thus we are able to regain in a unified and geometric way, the standard lifts of vectors and rank-2 tensors via the concept of ATL’s. Using the general Lie bracket relation (3.11), we can easily regain the known Lie brackets [1]–[4] amongst the three standard vector lifts:

$$[\overline{Y}, \overline{Z}] = \overline{[Y, Z]} - R(\widehat{Y}, Z), \quad [\overline{Y}, \widehat{Z}] = \widehat{\nabla_Y Z}, \quad (3.17)$$

$$[\overline{Y}, \widetilde{Z}] = \overline{[Y, Z]} + S(\widehat{Y}, Z), \quad [\widehat{Y}, \widehat{Z}] = 0, \quad (3.18)$$

$$[\widehat{Y}, \widetilde{Z}] = \widehat{[Y, Z]}, \quad [\widetilde{Y}, \widetilde{Z}] = \widetilde{[Y, Z]}, \quad (3.19)$$

where  $S(Y, Z)^a{}_b = (\mathcal{L}_Z \Gamma^a{}_{cb}) Y^c$ . Note that the sets of vertical and complete lifts each form a Lie algebra, but the horizontal lifts do not on a curved manifold. By (3.11), the vertical lifts form an ideal in the algebra of ATL’s, but the complete lifts do not.

We now show [7] that the LTL’s also include the matter symmetry vector fields of RKT. Berezdivin and Sachs define a matter symmetry as a vector field on  $TM$  that leaves the distribution function  $f$  unchanged. This vector field connects points in  $TM$  where the distribution of matter is the same. Geometrically, this implies that an observer at  $x$  with local Lorentz frame  $F$  will measure  $f$  on the tangent fibre  $T_x M$  to be the same as an observer at  $x'$  with Lorentz frame  $F'$  measuring  $f'$  on  $T_{x'} M$ . Thus matter symmetries arise in the class of LTL’s out of the requirement that the transport rule  $\Lambda$  be *Lorentz transport* along  $Y$ . Hence any vector transforms according to a representation of the Lorentz group  $SO(1, 3)$  along  $Y$ . Given an orthonormal tetrad  $\{E_a\}$ , we have  $E_a \cdot E_b = \eta_{ab} \equiv \text{diag}(-1, 1, 1, 1)$ . Now the tetrad components of any vector transform as  $u'^a = \Lambda^a(u, x; \varepsilon) = \Omega^a{}_b(x; \varepsilon) u^b$  where  $\Omega \in SO(1, 3)$ . Thus  $\Omega$  preserves  $\eta$ . Differentiating and noting that  $\Omega^a{}_b(x; 0) = \delta^a{}_b$ , we get

$$\omega_{(ab)} = 0 \Rightarrow A_{(ab)} = 0, \quad (3.20)$$

where  $\omega$  is defined by (3.6). This is the condition in (2.12) for  $Y^{(A)}$  to be a matter symmetry – or “Lorentz lift”. The matter symmetries form a Lie algebra, since by (3.12),  $C$  is skew if  $A$  and  $B$  are.

Iwai’s lift (2.9) arises as the LTL which is the lift of *conformal Lie transport*. However Iwai defines his lift for  $Y$  a projective collineation or conformal Killing vector, whereas the class of ATL’s generalises this to any  $Y$ :

$$Y^\dagger = Y^{(\nabla Y - 2\psi\delta)}. \quad (3.21)$$

The generalised Iwai lifts form a Lie algebra:

$$[Y^\dagger, Z^\dagger] = [Y, Z]^\dagger \quad \text{where} \quad \psi_{[Y, Z]} = \mathcal{L}_Y \psi_Z - \mathcal{L}_Z \psi_Y.$$

This generalises Iwai’s result [5] to the case of arbitrary  $Y, Z$ .

## 4 Dynamical and matter symmetries

In searching for a dynamical symmetry  $\Sigma$  obeying the condition (2.13) with  $\Gamma$  the geodesic spray (2.5), it is usually assumed that  $\Sigma$  arises purely from a vector field on the base manifold  $M$  – for example,  $\Sigma = \tilde{Y}$  or  $Y^\dagger$ . Transport lifts open up the possibility of generalising dynamical symmetries to the case where not only a vector field, but also a transport law for tangent vectors, is used to generate transformations of the dynamical trajectories. In the case of affine transport laws, this means looking at the ATL's. Unfortunately, as we shall show, the dynamical symmetry condition reduces the ATL to a vector lift – in fact to  $Y^\dagger$  [8]. At least this gives a foundation to the ad hoc ansatz of Iwai.

We examine now the conditions under which an ATL is a dynamical symmetry. By (2.13) this gives

$$[Y^{(A,k)}, \Gamma] = -\psi\Gamma, \quad (4.1)$$

where  $\Gamma$  is given by (2.5). Then (4.1) implies  $k^a = 0$ , which is the restriction to the class of LTL's. A further implication of (4.1) is that

$$A_{ab} = \nabla_b Y_a - \psi g_{ab}. \quad (4.2)$$

From (4.2) it is clear that  $\psi$  is restricted to  $\psi = \psi(x)$ , and

$$\mathcal{L}_Y \Gamma^a_{bc} \equiv \nabla_c \nabla_b Y^a - R^a_{bcd} Y^d = \delta^a_{(b} \nabla_{c)} \psi. \quad (4.3)$$

By (4.3),  $Y$  is a projective collineation vector [2], [3], and together with (4.2) this means that the ATL is reduced to Iwai's projective lift (2.9):

$$[Y^{(A,k)}, \Gamma] = -\psi\Gamma \Rightarrow Y^{(A,k)} = Y^{(\nabla Y - \psi\delta, 0)} = Y^\dagger.$$

Thus we see that any affinely based dynamical symmetry arises from a projective collineation vector. Furthermore, the ansatz introduced by Iwai in fact arises as the condition for an ATL to be a dynamical symmetry. Any attempt to generalise Iwai's ansatz would require a fully nonlinear transport rule  $\Lambda$ .

Matter symmetries provide a different, and more physically based, approach to symmetries of particle motion, but are correspondingly more difficult to analyse. By (2.12), we find that [7]

$$[Y^{(A)}, \Gamma] = (A^a_b - \nabla_b Y^a) p^b H_a + (R^a_{bcd} Y^d - \nabla_c A^a_b) p^b p^c V_a. \quad (4.4)$$

Thus a matter symmetry is more general than a dynamical symmetry, and reduces to the latter only if

$$A_{ab} = \nabla_{[b} Y_{a]}, \quad \nabla_{(b} Y_{a)} = \psi g_{ab}, \quad \mathcal{L}_Y \Gamma^a_{bc} = 0. \quad (4.5)$$

These conditions imply that  $Y$  is a conformal Killing and affine collineation vector, i.e. a homothetic vector ( $\psi$  is constant), and  $A$  is the bivector  $dY$ .

## 5 Conclusion

By generalising the concept of lifting point transformations to include tangent vector transport, we have defined the class of ATL's on the tangent bundle. The ATL's include all previous lifts, thus unifying many results into a single framework, with clear geometric interpretations. The generalisation introduced by the ATL concept includes in particular the matter symmetries of RKT, and the lifts introduced ad hoc by Iwai. The projective lift of Iwai is shown to be the unique ATL which is a dynamical symmetry on (semi-) Riemannian manifolds. The matter symmetries provide a very different concept of invariance – see [7] for a full discussion. They coincide with dynamical symmetries only in the special case that  $Y$  is homothetic and  $A = dY$ .

Applications of the ATL formalism beyond RKT are possible. It may also be useful in the study of symmetries in gauge field theories, since  $Y^{(A)}$  generates gauge transformations along  $Y$  if  $A$  is in the gauge Lie algebra at each point. The formalism could also be generalised to other fibre bundles. For example, an ATL on the  $\binom{r}{s}$  tensor bundle  $T^r_s M$  arises when  $\Lambda$  transforms  $\binom{r}{s}$  tensors along  $Y$ . With modifications, the formalism would also carry through to the tangent bundle of a manifold with torsion.

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